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Statistics of a Random Plus Constant Vector with Application to Acoustic and Electromagnetic-Wave Scattering

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A two-dimensional random-walk problem is investigated, where a vector fixed in amplitude (magnitude) is added to one that is random in phase (direction angle). This problem can arise from a study of the effect of multipath interference from an acoustic or electromagnetic-wave source, where an arrival of constant amplitude is combined with an incoherent background of randomly scattered components of the incident field. The source is considered to have harmonic time dependence. This study follows from an earlier paper, in which the required properties of the incoherent background were derived. In the present study, the joint distribution of the resultant amplitude and of the phase angle between the resultant and the fixed arrival is determined. Further, expressions are developed for the first and second moments of the resultant phase, amplitude, decibel-amplitude, and intensity. Many properties of these moments are described.

KEY WORDS: Propagation; acoustics; electromagnetic waves; multipath interference; scattering; incoherence; random walk.

1. INTRODUCTION

In a previous study,[7] the acoustic or electromagnetic-wave field scattered by a stochastic boundary, or by a random configuration of objects in space, was considered. In particular, conditions were examined for the scattered field to be incoherent.

The total field at a receiving point may result from the scattered and incident

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fields combining in phase interference. This paper investigates the total field when the scattered field is incoherent. The incident radiation is considered to have harmonic time dependence with circular frequency ω.

At a point in space and at a time τ, the randomly scattered field has the value

$$A_R \exp[i(\Phi_R' + \omega\tau)], \qquad A_R \geqslant 0, \quad -\pi \leqslant \Phi_R' < \pi \tag{1}$$

where A_R and Φ_R' denote its random amplitude and phase, respectively. The incident field at the same point and time has the value

$$A_o \exp[i(\Phi_o + \omega\tau)], \qquad A_o \geqslant 0 \tag{2}$$

where the amplitude A_o and phase Φ_o are not random and depend on the position of the observation point in space. Therefore, the total field there has amplitude A and phase Φ', which are given by

$$Ae^{i\Phi'} = A_o \exp(i\Phi_o) + A_R \exp(i\Phi_R'), \qquad A \geqslant 0, \quad -\pi \leqslant \Phi' < \pi \tag{3}$$

where the factor $e^{i\omega\tau}$ has been suppressed in all terms in the equation. If the terms in Eq. (3) are considered to be vectors in a plane, then the total field is represented by a vector sum of two components: the vector $A_o \exp(i\Phi_o)$ is constant in both its length A_o and its direction Φ_o, and is called the *fixed component*; the vector $A_R \exp(i\Phi_R')$ is random in both length and direction, and is called the *random component*. Therefore, the determination of the probability distribution of the resultant vector $Ae^{i\Phi'}$ is a problem in random walk.

Since $e^{i\Phi'}$ is a periodic function, a unique determination of Φ' from Eq. (3) requires the restriction of Φ' to some interval of length 2π rad. As observed in the above equation, the interval $[-\pi, \pi)$ is selected. However, this choice is quite arbitrary and, as we will observe shortly, is subject to change.

The joint probability density function of the random variables A_R and Φ_R' is denoted by $t(a_R, \phi_R)$. The assumption of an incoherent, random component has been shown to imply[7]

$$t(a_R, \phi_R) = u(\phi_R)\, g_T(a_R) \tag{4}$$

where $u(\phi_R)$ is the probability density of a random variable uniformly distributed on the interval $[-\pi, \pi)$, so that

$$u(\phi_R) = \begin{cases} (2\pi)^{-1} & \text{for} \quad -\pi \leqslant \phi_R < \pi \\ 0 & \text{for} \quad \phi_R \quad \text{elsewhere} \end{cases} \tag{5}$$

The function $g_T(a_R)$ is the probability density of A_R, and is studied in more detail in Ref. 7. Since $A_R \geqslant 0$, it should also be noted that

$$g_T(a_R) = 0 \qquad \text{for} \quad a_R < 0 \tag{6}$$

If the real and the imaginary parts of $A_R \exp(i\Phi_R')$ are jointly Gaussian-distributed, regardless of whether the random component is incoherent, then the probability distributions of A and Φ' are known.[3,5] If, in addition to Eq. (4), $g_T(a_R)$

is the probability density for the Rayleigh distribution, then A has the Rice–Nakagami distribution.[1] In this paper, however, we shall assume only the validity of Eq. (4).

Suppose now that Φ_o is not constant, but fluctuates as some stochastic or deterministic function of time. Multiplying Eq. (3) by $\exp(-i\Phi_o)$ and noting the periodicity of the function e^{ix}, we obtain

$$Ae^{i\Phi} = A_o + A_R \exp(i\Phi_R), \qquad -\pi \leqslant \Phi < \pi, \quad -\pi \leqslant \Phi_R < \pi \tag{7}$$

where[3]

$$\Phi \equiv \Phi' - \Phi_o \pmod{2\pi}, \qquad \Phi_R \equiv \Phi_R' - \Phi_o \pmod{2\pi} \tag{8}$$

If Φ_R' is independent of Φ_o and uniformly distributed on the interval $[-\pi, \pi)$, then Φ_R has been shown to be uniformly distributed on the same interval.[6] Furthermore, if A_R is independent of the vector (Φ_R', Φ_o), then A_R can be shown to be also independent of Φ_R. Therefore, the joint probability density of A_R and Φ_R is given by Eq. (4). Since Eq. (7) is identical to Eq. (3) (with $\Phi_o = 0$), the random-walk problem having a fluctuating Φ_o is reduced to the earlier one for which Φ_o is constant.

The problem of a fluctuating Φ_o may have application to situations where the radiation reaches the receiving point by way of two different paths of propagation. Consider, for example, an earth-bound radio transmitter. If the receiving point is near the horizon of the transmitting antenna, then the signal may be received by tropospheric scattering as well as by a path optically refracted through the atmosphere nearer to the earth's surface.[2] The scattered field may be incoherent, and the refracted signal can be constant in amplitude but may vary in phase. Since tropospheric scattering results from a process which is different from that governing the refracted path, the phase Φ_o of the refracted arrival can be expected to fluctuate independently of the amplitude A_R and phase Φ_R' of the scattered return. Therefore, using Eq. (4), the random variables A_R, Φ_R', and Φ_o are at least pairwise independent. If they can be considered totally independent, then the method outlined in the preceding paragraph applies.

In this paper, the phase difference Φ will be considered, rather than the resultant phase Φ'. If Φ_o fluctuates and is known statistically or deterministically, then Eq. (8) can be used to relate the statistical behavior of Φ' to that of Φ. Otherwise, such knowledge of Φ' cannot be obtained, so that the distribution of A is the only useful part of the solution to Eq. (7). If Φ_o is constant, then it is perhaps immaterial whether one examines Φ' or Φ, since the phase reference is quite arbitrary as long as it remains fixed with respect to the phase of the radiation's source.[4]

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[3] Mod 2π indicates that Φ is taken as the difference $\Phi' - \Phi_o$ mapped into the interval $[-\pi, \pi)$ by adding or subtracting the necessary integral multiple of 2π from any value of the difference. Similarly for Φ_R.

[4] In order to avoid mathematical difficulties in this last case, Φ' should be restricted to the interval $-\pi + \Phi_o \leqslant \Phi' < \pi + \Phi_o$ instead of the interval exhibited in Eq. (3). The first equation in Eq. (8) can then be changed to an ordinary equation with the mod 2π notation omitted. This practice assures that the graph for the probability density of Φ' is a single, horizontal translation of that for Φ through a distance of Φ_o. Otherwise, a more complicated relationship between the probability densities would exist, which would be as artificial from a physical point of view as the choice of the interval for Φ'.

The strength of an acoustic or electromagnetic-wave field is commonly measured in any of three units: the intensity, the amplitude, and the decibel-amplitude. In this paper, all three measures will be examined for the total field. The properties of their mean values, and also of their standard deviations, can be quite different, as we will demonstrate.

In Section 2 of this paper, the joint probability density of A and Φ is derived. Here, the mean value of Φ and the correlation between Φ and $f(A)$ are determined, where f represents any sufficiently integrable function. The mean values and standard deviations of A and A^2, the latter being proportional to the intensity of the total field, are derived in Section 3. In Section 4, the mean value of $20 \log_{10} A$, which is the resultant decibel-amplitude, is examined, as are the standard deviations of Φ and $20 \log_{10} A$. A summary follows in Section 5.

2. THE JOINT DISTRIBUTION OF A AND Φ

Figure 1 is a vector diagram illustrating Eq. (7). One observes that a transformation exists between the random variables (A_R, Φ_R) and the variables (A, Φ). In particular, it can be shown from Eq. (7) that

$$\begin{aligned} A_R = \mathcal{A}_R(A, \Phi) &\equiv |(A \cos \Phi - A_o) + iA \sin \Phi | \\ &= (A_o{}^2 - 2A_oA \cos \Phi + A^2)^{1/2} \end{aligned} \tag{9a}$$

$$\begin{aligned} \Phi_R = \mathcal{P}_R(A, \Phi) &\equiv \arg[(A \cos \Phi - A_0) + iA \sin \Phi] \\ &= \begin{cases} -\text{Arc}\cos[(A \cos \Phi - A_o)(A_o{}^2 - 2A_oA \cos \Phi + A^2)^{-1/2}] & \text{for } -\pi \leqslant \Phi < 0, \\ \text{Arc}\cos[(A \cos \Phi - A_o)(A_o{}^2 - 2A_oA \cos \Phi + A^2)^{-1/2}] & \text{for } 0 \leqslant \Phi < \pi \end{cases} \end{aligned} \tag{9b}$$

Therefore, the joint probability density of A and Φ, which is denoted by $p(a, \phi)$, is given by

$$p(a, \phi) = t[\mathcal{A}_R(a, \phi), \mathcal{P}_R(a, \phi)] \mid J(a, \phi)| \tag{10}$$

where J is the Jacobian of the transformation in Eq. (9). It can be shown that[5]

$$J(a, \phi) \equiv \frac{\partial(\mathcal{A}_R, \mathcal{P}_R)}{\partial(a, \phi)} = \frac{a}{\mathcal{A}_R(a, \phi)} \tag{11}$$

[5] In deriving Eq. (11), it is convenient to treat the transformation in Eq. (9) as a system of two equations: $A_R \cos \Phi_R = A \cos \Phi - A_0$ and $A_R \sin \Phi_R = A \sin \Phi$.

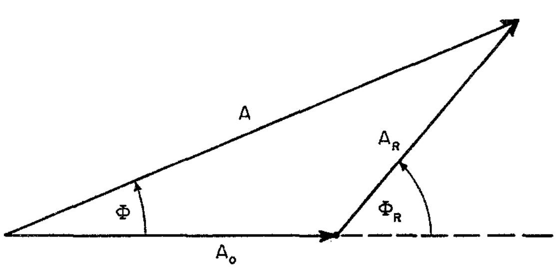


Fig. 1. Vector diagram of random-walk problem.

From Eqs. (4), (5), and (9)–(11), we obtain

$$p(a, \phi) = \begin{cases} \dfrac{a g_T[(A_o^2 - 2A_o a \cos\phi + a^2)^{1/2}]}{2\pi(A_o^2 - 2A_o a \cos\phi + a^2)^{1/2}} & \text{for} \quad a \geqslant 0, \quad -\pi \leqslant \phi < \pi, \\ 0 & \text{for} \quad a \text{ or } \phi \text{ elsewhere} \end{cases} \tag{12}$$

From Eq. (12), $p(a, \phi)$ is observed to be an even function of ϕ. Therefore,

$$E(\Phi) = \int_0^\infty \int_{-\pi}^{\pi} \phi p(a, \phi)\, d\phi\, da = 0 \tag{13}$$

where E denotes mathematical expectation. More generally,

$$E[\Phi f(A)] = \int_0^\infty f(a) \Big[\int_{-\pi}^{\pi} \phi p(a, \phi)\, d\phi \Big]\, da = 0 \tag{14}$$

where f is any function for which the integrand in Eq. (14) is absolutely integrable. Since Φ is the phase difference between the fixed and random components (recall footnote 4), Eq. (13) asserts that the mean resultant phase is just the phase of the fixed component, when Φ_o is constant.

Equation (12) shows that the random variables A and Φ are generally dependent. However, from Eqs. (13) and (14), we obtain

$$E[\Phi f(A)] - E(\Phi)\, E[f(A)] = 0 \tag{15}$$

Therefore, the phase angle Φ is uncorrelated with $f(A)$. In particular, we may wish to take $f(A)$ as the resultant decibel-amplitude $20 \log_{10} A$, the resultant intensity (which is proportional to A^2), or just the resultant amplitude A itself. If Φ_o is constant, it is not difficult to show that Eq. (15) is also valid if the resultant phase Φ' is substituted for Φ (recall footnote 4).

In the remainder of this paper, the first and second moments of A, of A^2, and of $20 \log_{10} A$, and the second moment of Φ are examined in terms of the distribution of A_R. Although these moments may be derived from Eq. (12), it is more convenient to make use of Eq. (4).

3. THE RESULTANT AMPLITUDE AND INTENSITY

One observes that A is a function of the random variables A_R and Φ_R, which we shall denote by $\mathscr{A}(A_R, \Phi_R)$. Similarly, we denote the dependence of Φ by the function $\mathscr{P}(A_R, \Phi_R)$. From Eq. (7), $\mathscr{A}$ and $\mathscr{P}$ can be shown to be given by

$$\mathscr{A}(a_R, \phi_R) = | A_o + a_R \exp(i\phi_R)| \tag{16a}$$

$$\mathscr{P}(a_R, \phi_R) = \arg[A_o + a_R \exp(i\phi_R)] \tag{16b}$$

From Eq. (16a),

$$E(A) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t(a_R, \phi_R)\, \mathcal{A}(a_R, \phi_R)\, d\phi_R\, da_R \tag{17}$$

From Eqs. (4)–(6) and (17), the mean amplitude of the total field is given by

$$E(A) = \int_0^{\infty} g_T(a_R)\Big[(2\pi)^{-1} \int_{-\pi}^{\pi} \mathcal{A}(a_R, \phi_R)\, d\phi_R\Big]\, da_R \tag{18}$$

Using Eq. (16a), it can be shown that

$$\begin{aligned}\mathcal{A}(a_R, \phi_R) &= (A_o^2 + 2A_o a_R \cos\phi_R + a_R^2)^{1/2} \\ &= | A_o + a_R | \,[1 - 4A_o a_R(A_o + a_R)^{-2} \sin^2 \tfrac{1}{2}\phi_R]^{1/2}\end{aligned} \tag{19}$$

Multiplying Eq. (19) by $(2\pi)^{-1}$ and integrating with respect to ϕ_R, it is observed after simplifying that

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \mathcal{A}(a_R, \phi_R)\, d\phi_R = (2/\pi)| A_o + a_R |\, \mathscr{E}(2A_o^{1/2}a_R^{1/2} | A_o + a_R |^{-1}) \tag{20}$$

where $\mathscr{E}(x)$ is the complete elliptic integral of the second kind.[4] From Eqs. (18) and (20), we obtain

$$E(A) = \int_0^{\infty} g_T(a_R)(A_o + a_R)(2/\pi)\, \mathscr{E}[2(A_o a_R)^{1/2} (A_o + a_R)^{-1}]\, da_R \tag{21}$$

Tables and approximations of $\mathscr{E}(x)$ are available,[4] so that $E(A)$ can be obtained from Eq. (21) if $g_T(a_R)$ is specified.

From Eq. (19), or using the law of cosines on Fig. 1, we have

$$A^2 = A_o^2 + 2A_o A_R \cos \Phi_R + A_R^2 \tag{22}$$

Recalling that A_R and Φ_R are independent random variables and noting that $E(\cos \Phi_R) = 0$ [see Eqs. (4) and (5)], we obtain from Eq. (22)

$$E(A^2) = A_o^2 + E(A_R^2) \tag{23}$$

The variance of the resultant amplitude is then obtained by subtracting the square of Eq. (21) from Eq. (23).

Since intensity is proportional to amplitude squared, Eq. (23) serves to determine the mean intensity of the total field. In particular, the mean value of the resultant intensity is observed to be equal to the sum of the mean intensities of both components. By squaring Eq. (22) and taking the mean value of both sides of the resulting equation, we obtain

$$E(A^4) = A_o^4 + 4A_o^2 E(A_R^2) + E(A_R^4) \tag{24}$$

where it is noted that $E(\cos^2 \Phi_R) = \frac{1}{2}$. Then, subtracting the square of Eq. (23) from Eq. (24), the variance of the resultant intensity is found to be proportional to

$$\sigma^2(A^2) = \sigma^2(A_R^2) + 2A_o^2 E(A_R^2) \tag{25}$$

where σ^2 denotes variance.

4. RESULTANT DECIBEL-AMPLITUDE AND PHASE

From Eqs. (4)–(6), the mean of ln A is given by

$$\begin{aligned} E(\ln A) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t(a_R, \phi_R) \ln \mathscr{A}(a_R, \phi_R)\, d\phi_R\, da_R \\ &= \int_0^{\infty} g_T(a_R)\left[(2\pi)^{-1} \int_{-\pi}^{\pi} \ln \mathscr{A}(a_R, \phi_R)\, d\phi_R\right] da_R \end{aligned} \tag{26a}$$

Similarly, the mean squares of ln A and Φ are given by

$$E[(\ln A)^2] = \int_0^{\infty} g_T(a_R)\left\{(2\pi)^{-1} \int_{-\pi}^{\pi} [\ln \mathscr{A}(a_R, \phi_R)]^2\, d\phi_R\right\} da_R \tag{26b}$$

$$E(\Phi^2) = \int_0^{\infty} g_T(a_R)\left\{(2\pi)^{-1} \int_{-\pi}^{\pi} [\mathscr{P}(a_R, \phi_R)]^2\, d\phi_R\right\} da_R \tag{26c}$$

Subtracting the square of Eq. (26a) from Eq. (26b) yields the variance of ln A. Similarly, the variance of Φ is obtained from Eqs. (13) and (26c). The moments of the resultant decibel-amplitude are obtained from those of ln A by noting that

$$20 \log_{10} A = (20/\ln 10) \ln A \tag{27}$$

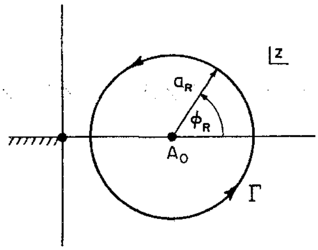
We now begin the treatment of the iterated integrals in Eq. (26) by first considering the integrations with respect to ϕ_R. These integrals are determined by examining contour integrations of the function $(\log z)/(z - A_o)$ and $(\log z)^2/(z - A_o)$ in the complex plane. Only that branch of the complex logarithm function is considered for which $-\pi \leqslant \mathrm{Im}(\log z) < \pi$, where Im denotes imaginary part.

Figure 2 defines the contour Γ in the complex plane. When $a_R < A_o$, Γ is just the circle $|z - A_o| = a_R$, as shown in Fig. 2(a). However, when $a_R > A_o$, Γ is the contour illustrated in Fig. 2(b), where the branch cut for the function log z is accounted for. In either case, Cauchy's integral formula implies

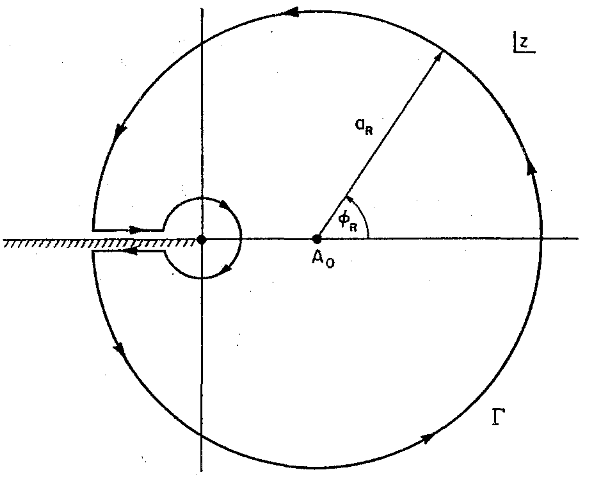
$$\int_\Gamma (\log z)/(z - A_o)\, dz = 2\pi i \log A_0 \tag{28a}$$

$$\int_\Gamma (\log z)^2/(z - A_o)\, dz = 2\pi i(\log A_o)^2 \tag{28b}$$

The contribution to the integrals in Eq. (28) from the inner circle in Fig. 2(b) can be shown to vanish in the limit as its radius goes to zero. The integrals along the line segments connecting the inner circle to the outer one are expressed in Riemann



(a) $a_R < A_0$



(b) $a_R > A_0$

Fig. 2. Contours of integration in the complex z-plane.

integrals, noting that $\text{Im}(\log z) = \arg z \equiv \pi$ on the upper segment and $-\pi$ on the lower one. The integrals along the outer circle in Fig. 2(b), or along the circle in Fig. 2(a), can be shown to be given by

$$\int_{|z-A_o|=a_R} (\log z)/(z - A_o)\, dz$$

$$= i \int_{-\pi}^{\pi} \ln \mathscr{A}(a_R, \phi_R)\, d\phi_R - \int_{-\pi}^{\pi} \mathscr{P}(a_R, \phi_R)\, d\phi_R \tag{29a}$$

$$\int_{|z-A_o|=a_R} (\log z)^2/(z - A_o)\, dz$$

$$= i \left\{ \int_{-\pi}^{\pi} [\ln \mathscr{A}(a_R, \phi_R)]^2\, d\phi_R - \int_{-\pi}^{\pi} [\mathscr{P}(a_R, \phi_R)]^2\, d\phi_R \right\}$$

$$- 2 \int_{-\pi}^{\pi} \mathscr{P}(a_R, \phi_R) \ln \mathscr{A}(a_R, \phi_R)\, d\phi_R \tag{29b}$$

Equation (29) is derived from the relation

$$\log z = \ln \mathscr{A}(a_R, \phi_R) + i\mathscr{P}(a_R, \phi_R), \qquad |z - A_o| = a_R \tag{30}$$

which is valid for all points $z = A_o + a_R \exp(i\phi_R)$ on the circle $|z - A_o| = a_R$, and which follows from Eq. (16).

Equating the imaginary parts of both sides of each of the equations resulting from the procedure outlined above, Eqs. (28a) and (28b) yield, respectively,

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \ln \mathscr{A}(a_R, \phi_R)\, d\phi_R = \begin{cases} \ln A_o & \text{for} \quad a_R < A_o \\ \ln a_R & \text{for} \quad a_R > A_o \end{cases} \tag{31a}$$

$$(2\pi)^{-1} \int_{-\pi}^{\pi} [\ln \mathscr{A}(a_R, \phi_R)]^2\, d\phi_R - (2\pi)^{-1} \int_{-\pi}^{\pi} [\mathscr{P}(a_R, \phi_R)]^2\, d\phi_R - (\ln A_o)^2$$
$$= \begin{cases} 0 & \text{for} \quad a_R < A_o \\ 2 \displaystyle\int_{A_o}^{a_R} x^{-1} \ln(x - A_o)\, dx & \text{for} \quad a_R > A_o \end{cases} \tag{31b}$$

Multiplying Eq. (31a) by $g_T(a_R)$, integrating with respect to a_R, and using Eqs. (26a) and (27), the mean decibel-amplitude is found to be

$$E(20 \log_{10} A) = (20 \log_{10} A_0) \int_0^{A_o} g_T(a_R)\, da_R + (20/\ln 10) \int_{A_o}^{\infty} g_T(a_R) \ln a_R\, da_R \tag{32}$$

Using this last equation, the mean decibel-amplitude can be shown to be given by either of the following two equations:

$$E(20 \log_{10} A) = 20 \log_{10} A_o + (20/\ln 10) \int_{A_o}^{\infty} g_T(a_R) \ln(a_R/A_o)\, da_R \tag{33a}$$

$$E(20 \log_{10} A) = E(20 \log_{10} A_R) + (20/\ln 10) \int_0^{A_o} g_T(a_R) \ln(A_o/a_R)\, da_R \tag{33b}$$

Multiplying Eq. (31b) by $(20/\ln 10)^2 g_T(a_R)$ and integrating with respect to a_R, we obtain, using Eqs. (26b) and (26c),

$$E[(20 \log_{10} A)^2] - (20 \log_{10} A_o)^2 - (20/\ln 10)^2 E(\Phi^2)$$
$$= 2(20/\ln 10)^2 \int_{A_o}^{\infty} g_T(a_R) \left[\int_{A_o}^{a_R} x^{-1} \ln(x - A_o)\, dx \right] da_R \tag{34}$$

Equation (34) relates the second moments of Φ and of $20 \log_{10} A$, although it does not evaluate them. Equations (33) and (34) will be examined next.

Observe that the last term in Eq. (33b) is independent of $g_T(a_R)$ for those values of $a_R > A_o$. Similarly, Eqs. (33a) and (34) appear to be independent of $g_T(a_R)$ for values of $a_R < A_o$. In order to illustrate the significance of these observations, two examples shall now be considered. In the first example, the amplitude A_R of the random component is assumed to be always larger than A_o of the fixed component. Therefore,

$g_T(a_R)$ vanishes for $a_R \leqslant A_o$ and is otherwise arbitrary as a probability density, provided that the moments and integrals involved in the relevant equations exist. From Eq. (33b), we obtain[6]

$$E(20 \log_{10} A) = E(20 \log_{10} A_R), \qquad \text{prob}(A_R > A_o) = 1 \tag{35}$$

In the second example, A_R is assumed to always be smaller than A_o. From Eq. (33a), we have

$$E(20 \log_{10} A) = 20 \log_{10} A_o, \qquad \text{prob}(A_R < A_o) = 1 \tag{36}$$

Therefore, the mean decibel-amplitude of the resultant is equal to the decibel-amplitude of the fixed component, and is independent of the distribution of A_R. Furthermore, the right side of Eq. (34) is observed to vanish; and, using Eqs. (13) and (36), it can be shown that

$$\sigma(20 \log_{10} A) = (20/\ln 10)\, \sigma(\Phi), \qquad \text{prob}(A_R < A_o) = 1 \tag{37}$$

where σ denotes standard deviation. If Φ_o is constant, then Φ' may replace Φ in the above equation. Therefore, the standard deviation of the resultant phase is just proportional to that of the resultant decibel amplitude. However, nothing quite as simple can be concluded from Eq. (34) when A_R is always larger than A_o.

If the event $A_R > A_o$ is sufficiently improbable, then Eqs. (36) and (37) may serve as approximations, whose errors can be determined from the rightmost terms in Eqs. (33a) and (34). Similarly, Eq. (35) may approximate the more complicated relationship in Eq. (33b) if the event $A_R < A_o$ is sufficiently improbable.

Equation (37) may be useful in determining the standard deviation of the received phase when only the resultant amplitude has been recorded in a propagation experiment. However, evidence would have to be present to indicate that prob$(A_R < A_o)$ is equal to unity or sufficiently close to it. Equations (35) and (36) may be used to demonstrate some properties of a logarithmic amplifier when its output is averaged. In particular, the "noise" due to the randomly scattered arrivals is eliminated in the averaged output if A_R does not exceed A_o. However, only the noise will be heard if A_R is always larger than A_o.

It may be noted that the integrands of the rightmost terms in Eqs. (33a) and (33b) are nonnegative over the respective intervals of integration, so that the terms themselves are nonnegative. Therefore, the mean decibel-amplitude of the total field is observed to be at least as large as that of either component.

5. SUMMARY

In this paper, an acoustic or electromagnetic-wave field is considered to contain just two components. The fixed component is taken to be constant in amplitude, while the random component is the result of incoherent scattering of the incident field.

[6] The notation prob$(A_R > A_o)$ denotes the probability of the event $A_R > A_o$, so that the statement to the right of Eq. (35) indicates the condition under which the equation was derived.

The joint probability density of the phase angle Φ and the amplitude of the total field is derived in terms of the probability density of the amplitude of the random component. Although the resultant amplitude and angle Φ are observed to be dependent random variables, they are shown to have zero correlation.

The first and second moments of the resultant intensity, the resultant amplitude, and the resultant decibel-amplitude are observed to behave somewhat differently from one another. It is known, for example, that the mean intensity of the total field is equal to the sum of the mean intensities of the components. However, if one component is always larger than the second, then the mean decibel-amplitude of the total field is shown to be equal to that of the larger component, and is independent of the smaller one. The mean and standard deviation of the resultant amplitude is given in terms of the amplitude of the fixed component, the probability density of the amplitude of the random component, and a complete elliptic integral of the second kind.

If the phase of the fixed component is a constant, then it is observed to be equal to the mean value of the resultant phase. If, in addition, the random component is always smaller than the fixed component, then the standard deviations of the resultant phase and the resultant decibel-amplitude are found to be proportional.

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